

Freezing similarity solutions in multi-dimensional Burgers' Equation

Jens Rottmann-Matthes¹

Institut für Analysis
Karlsruhe Institute of Technology
76131 Karlsruhe
Germany

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Abstract. The topic of this paper are similarity solutions occurring in multi-dimensional Burgers' equation. We present a simple derivation of the symmetries appearing in a family of generalizations of Burgers' equation in d -space dimensions. These symmetries we use to derive an equivalent partial differential algebraic equation (freezing system) that allows us to do long time simulations and obtain good approximations of similarity solutions by direct forward simulation. The method also allows us without further effort to observe meta-stable behavior near N-wave-like patterns.

Key words. Similarity solutions, relative equilibria, Burgers' equation, freezing method, scaling symmetry, meta-stable behavior.

AMS subject classification. 65P40, 35B40, 35B06, 37C80 (65M99, 35L65)

1. INTRODUCTION

Patterns are abundant in partial differential equations and they often have an important implication on the interpretation of the system's behavior. A simple type of pattern are relative equilibria and the possibly simplest, non-trivial relative equilibria in partial differential equations are traveling waves. Traveling waves for example appear in the case of the nerve-axon-equations (e.g. in the Hodgkin-Huxley system) and can be interpreted as the transport of information. From this interpretation, it is obvious, that one is not only interested in the waves shape, but also, how fast it actually travels and how it evolves in the first place, thus, how fast the information is actually passed on from its ignition.

One possibility to calculate this by a direct forward simulation, is the method of freezing, independently introduced in [5] and [20], see also [4]. The method not only allows to capture traveling waves by simple long-time simulations, but can also be used for other relative equilibria such as rotating waves or scroll waves. In this article we show how the method can be used to do long-time simulations of the multi-dimensional Burgers' equation and how similarity solutions of Burgers' equation can be obtained in this way. We do not consider the most general and abstract version of the method but only consider it in such generality as needed for the specific case of Burgers' equation. We refer to the dissertation [18] and also [4], which is based on [5] and [20] for more on the abstract method. But also note that as in [20] and unlike [5], we include a time-scaling. This will be crucial for long time simulations and the ability to observe meta-stable behavior in Burgers' equation.

Burgers' equation,

$$(1) \quad u_t + \left(\frac{1}{2}u^2\right)_x = \nu u_{xx}, \quad x \in \mathbb{R},$$

¹e-mail: jens.rottman-matthes@kit.edu, phone: +49 (0)721 608 41632,
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was originally introduced by J.M. Burgers (e.g. [6]) in 1948 as a model of turbulence. In [6] he points out that the combination of the dissipative term νu_{xx} and the nonlinear term uu_x characterizes the mechanism of producing turbulence. Equation (1) is one of the simplest truly nonlinear partial differential equations and it is well-known that it produces shock solutions in the inviscid case ($\nu = 0$). Therefore, it is also frequently used as a test equation for numerical schemes for conservation laws and for shock-capturing schemes. We also consider the following multi-dimensional generalizations of Burgers' equation,

$$(2) \quad u_t + \frac{1}{p} \operatorname{div}(a|u|^p) = \nu \Delta u, \quad x \in \mathbb{R}^d,$$

where $d \geq 1$, $a \in \mathbb{R}^d \setminus \{0\}$, $\nu > 0$ and $p > 1$ are fixed. The special case of $p = 2$ we call the d -dimensional Burgers' equation and we call the case $p = \frac{d+1}{d}$ the conservative Burgers' equation. Note that for $d = 1$ and $a = 1$ both special cases reduce to the standard Burgers' equation (1). Equation (2) with $p = 2$ appears as special cases of the multidimensional Burgers' equation

$$(3) \quad \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = \nu \Delta \vec{u},$$

which has applications in different areas of physics, see the review [2]. For example, if $\vec{u}_1 = u$ and $\vec{u}_j = 0$ for $j = 2, \dots, d$, (3) leads to (2) with $a = e_1$.

We use the symmetries inherent in (2) to split the evolution of the solution into a part that captures the evolution of the profile and a part that captures the evolution of the solution in the symmetry group. This transforms the Cauchy problem for the original partial differential equation (PDE) (2) into an equivalent partial differential algebraic equation (PDAE), the so-called freezing system. Stationary solutions to the freezing system PDAE are similarity solutions of (2), and they not only include the profile of these but also the full information about the evolution of this profile in the symmetry group. Moreover, if the similarity solutions are asymptotically stable, they can be obtained by a direct forward simulation of this PDAE system. This has been proved rigorously in several cases, e.g. see [19].

From an analytic point of view, the most interesting scale of viscosities ν in (1) is $\nu \approx 0$. In this region the solution to the Cauchy problem for (1) exhibits a metastable behavior in the sense, that it rapidly approaches a similarity solution of the inviscid problem, namely an N -wave and then has a very long transient until it finally reaches a true similarity solution of the parabolic problem, a so called viscosity wave. This has first been observed numerically and analyzed in [10] and from a dynamical systems point of view in [3] by means of the Cole-Hopf-transform. In both articles the authors have used correct asymptotic similarity variables, which can explicitly be calculated for Burgers' equation. In contrast to [10] and [3], we take the point of view that in more complicated equations these similarity variables are not known a priori and can only be obtained by a numerical calculation. In fact, our method does not need this information but actually calculates a suitable choice of similarity variables on the fly. The similarity variables obtained by our method may differ from the simple scaling variables used in [10] and [3]. This is because we do not center the solution at 0 and, moreover, not only allow scalings, but also a non-zero velocity in the similarity variables.

The plan of the paper is as follows. In Section 2 we use a transformation of the coordinates to write the generalized Burgers' equation (2) in a simple canonical form. For this simple form we then derive a continuous family of symmetries inherent in the equation which can be used for the numerical calculation. In Section 3 we show that the symmetry group obtained in Section 2 acts strongly continuous on various function spaces. This allows us to calculate the generators of the group action on suitable function spaces. In Section 4 we then make the ansatz that the evolution of the solution to (2) can be split up into an evolution of the profile and an evolution along the group orbit to derive the equation in a co-moving frame. This ansatz introduces new unknowns into the equation. For example in the case $d = 2$, $p = \frac{3}{2}$, and $a = e_1$ in (2) this leads to the following under-determined PDE

$$v_\tau = \nu \Delta v - v v_x + \mu_1((xv)_x + (yv)_y) + \mu_2 v_x + \mu_3 v_y$$

for v , μ_1 , μ_2 , and μ_3 . We present two possible choices of phase-conditions to cope with these artificially introduced degrees of freedom. The resulting equation is then the freezing PDAE and we show that it is equivalent to the original problem. In the final Section 5 we present the numerical results of several experiments. These show the ability of the method to calculate similarity solutions of the multi-dimensional Burgers' equation by direct forward simulation. With our method we are also able to observe directly a meta-stable behavior of the solution not only in the 1d-case as in [10], but also for the multi-dimensional Burgers' equation with small viscosity in the vicinity of so called N-wave like patterns. To our knowledge, this is the first time, this has been observed in the multi-dimensional case. Closest to our approach is the article [20] where a variant of the method was introduced for different examples, including the standard 1d Burgers' equation. But note that in that article neither the generalization to multi-dimensional problems or more general symmetries as in (2) were considered, nor the long-time or meta-stable behavior was observed.

Throughout this article we use the following notations and spaces. We denote the Euclidean norm in a finite dimensional space by $|\cdot|$ and inner products in \mathbb{R}^d we write as $\langle \cdot, \cdot \rangle$. For vectors $x \in \mathbb{R}^d$, x^\top denotes the transpose, so that $\langle x, y \rangle = x^\top y$ for all $x, y \in \mathbb{R}^d$. We interchangeably use the notations $\frac{\partial}{\partial x_j} u = \partial_{x_j} u = \partial_j u = u_{x_j}$ for the derivative with respect to x_j . By Df we denote the total derivative (or Jacobian) of a function f and $D^\alpha f$ denotes the partial derivative $\partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f$, where $\alpha \in \mathbb{N}^d$ is a multi-index. For differentiable maps between manifolds, $f : M \rightarrow N$, we denote by $T_{u_0} f : T_{u_0} M \rightarrow T_{f(u_0)} N$, $v \mapsto T_{u_0} f[v]$ the tangential of the map f at the point $u_0 \in M$.

Since we are primarily interested in the approximation of localized (similarity) solutions, we consider functions, vanishing at infinity in a suitable sense. Therefore, we consider the space of k -times continuously differentiable functions with compact support,

$$\mathcal{C}_c^k(\mathbb{R}^d) = \{u \in \mathcal{C}^k(\mathbb{R}^d) : \text{supp}(u) \subset \mathbb{R}^d \text{ is compact}\},$$

and the space of k -times continuously differentiable functions vanishing at infinity,

$$\mathcal{C}_0^k(\mathbb{R}^d) = \left\{ u \in \mathcal{C}^k(\mathbb{R}^d) : \lim_{|x| \rightarrow \infty} |D^\alpha u(x)| = 0 \ \forall \alpha \in \mathbb{N}^d, |\alpha| \leq k \right\}.$$

The space $\mathcal{C}_0^k(\mathbb{R}^d)$ with norm $\|u\|_{\mathcal{C}^k} := \max_{|\beta| \leq k} \|D^\beta u\|_\infty$ is a Banach space. Another suitable choice are the L^2 -Sobolev-spaces $H^k(\mathbb{R}^d)$ with the norm $\|u\|_{H^k}^2 = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2}^2$ which is a Hilbert space for each $k \in \mathbb{N}$. In $H^k(\mathbb{R}^d)$ we also use the equivalent norm

$$(4) \quad \|u\|_{\dot{H}^k} := \|\langle \xi \rangle^k \hat{u}\|_{L^2},$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$ and

$$\hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix^\top \xi} u(x) dx$$

is the Fourier transform of u (cf. [21, Ch.7]).

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2. CANONICAL FORM AND SYMMETRIES OF GENERALIZED BURGERS' EQUATION

In this section we first derive a standard form of the generalized Burgers' equation (2). Then we will calculate certain symmetries for this standard form. It is well-known and already appears in [9] that Burgers' equation (1) exhibits several symmetries. The system has often served as an example to illustrate similarity methods, e.g. see [13, Ex. 6.1] where a five-dimensional Lie algebra corresponding to the symmetries in Burgers' equation is calculated and also similarity solutions are obtained. Here we do not follow the abstract approach from [14] but rather directly supply suitable symmetries, which we will later on use in our numerical method. For example, we ignore the shift equivariance with respect to

time, which is of no use to us, since we are interested in the behavior of solutions to the corresponding Cauchy problems.

2.1. Canonical Form. To write equation (2) in a standard form, we use the following lemma.

Lemma 2.1. *Let $M \in \mathbb{R}^{d,d}$ be invertible and identify M with the linear mapping in \mathbb{R}^d it induces. Furthermore let $1 \leq p < \infty$.*

(1) *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz continuous, it follows for $u \in W^{1,p}(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d)$ with $f \circ u \in L^p$: $f \circ u \circ M \in W^{1,p}(\mathbb{R}^d)$ and*

$$(5) \quad D(f \circ u \circ M)(x)\xi = Df(u(Mx))Du(Mx)M\xi$$

for almost every $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d$.

(2) *If $u \in W^{2,p}(\mathbb{R}^d)$, then $u \circ M \in W^{2,p}(\mathbb{R}^d)$ and*

$$(6) \quad \Delta(u \circ M)(x) = \sum_{i,j,k} M_{ij} \partial_i \partial_k u(Mx) M_{kj} = \text{tr}(M^\top \text{Hess}(u)M)(Mx)$$

for almost every $x \in \mathbb{R}^d$.

The chain-rule immediately implies that (5), (6) hold for $u \in \mathcal{C}^2(\mathbb{R}^d)$. The L^p -case considered in Lemma 2.1 follows from [23, §2]. The above lemma shows that for $X = \mathcal{C}_0^0(\mathbb{R}^d)$ and $Y = \mathcal{C}_0^2(\mathbb{R}^d)$, respectively $X = L^2(\mathbb{R}^d)$ and $Y = H^2(\mathbb{R}^d)$, a function

$$u \in \mathcal{C}([0, T]; Y) \cap \mathcal{C}^1((0, T); X)$$

solves (2) if and only if

$$v \in \mathcal{C}([0, |a|T]; Y) \cap \mathcal{C}^1((0, |a|T); X),$$

given by

$$u(x, t) = v(Q_a x, \frac{1}{|a|}t) = v(y, s), \quad q = a - |a|e_1, \quad Q_a = I - 2\frac{q q^\top}{q^\top q},$$

satisfies

$$(7) \quad \frac{\partial}{\partial s} v + \frac{1}{p} \frac{\partial}{\partial y_1} (|v|^p) = \frac{\nu}{|a|} \Delta v.$$

Here we say that a function solves (2) or (7) if the equalities hold for all t as equalities in X . In the sequel we incorporate the factor $\frac{1}{|a|}$ into the viscosity ν , so that we restrict to

$$(8) \quad u_t = \nu \Delta u + \frac{1}{p} \frac{\partial}{\partial x_1} (|u|^p) =: F(u).$$

Remark 2.2. A simple well-posedness result can be found e.g. in [15, §8 Thm. 3.5] in the sense that for every initial data $u_0 \in H^2$ there exists a unique strong solution. See also [15, § 7.2.5].

From now on we do not discuss the well-posedness of the problems, but require that the solution exists and belongs to the function space at hand without further notice.

2.2. Symmetries for Burgers' Equation. To motivate the ansatz, we remark that the nonlinear operator $F : u \mapsto \nu \Delta u - \frac{1}{p} (|u|^p)_{x_1}$ is equivariant with respect to spatial translations in every direction and with respect to rotations that leave the first coordinate axis fixed. Moreover, each of the summands has a scaling property and we obtain a scaling symmetry by matching these. More precisely, we make the ansatz

$$(9) \quad u(x) = \frac{1}{\alpha} v(M^{-1}(x - b)),$$

where $\alpha > 0$, $b \in \mathbb{R}^d$, and $M \in \mathbb{R}^{d,d}$ is some invertible matrix. Using the chain rule, respectively Lemma 2.1, easily follows:

Proposition 2.3. *Let $p > 1$ be fixed. For all $u \in \mathcal{C}^2(\mathbb{R}^d)$ (resp. $u \in H^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$) and v given by (9), holds*

$$(10) \quad F(u) = \text{const}_{(\alpha, M, b)} \frac{1}{\alpha} F(v)(M^{-1}(\cdot - b))$$

as an equality in \mathcal{C}^0 (resp. L^2) if and only if $M = \alpha^{p-1} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$, where $Q \in \mathcal{O}(d-1)$, and $\text{const}_{(\alpha, M, b)} = \alpha^{2-2p}$.

Proof. The chain rule (resp. Lemma 2.1) implies for the left hand side of (10)

$$\begin{aligned} F(u)(x) &= \frac{1}{\alpha} \nu \text{tr}(\text{Hess}(v) M^{-1} M^{-\top})(M^{-1}(x - b)) \\ &\quad - \frac{1}{\alpha^p} |v(M^{-1}(x - b))|^{p-1} \text{sgn}(v(M^{-1}(x - b))) Dv(M^{-1}(x - b)) M^{-1} e_1 \end{aligned}$$

for all $x \in \mathbb{R}^d$ (respectively for almost every $x \in \mathbb{R}^d$ in the H^2 -case). Similarly, the right hand side of (10) equals for all $x \in \mathbb{R}^d$ (resp. for a.e. $x \in \mathbb{R}^d$)

$$\text{const}_{(\alpha, M, b)} \frac{1}{\alpha} \left(\nu \text{tr}(\text{Hess}(v)) + |v|^{p-1} \text{sgn}(v) \frac{\partial}{\partial x_1} v \right) (M^{-1}(x - b)).$$

Therefore, (10) holds as a pointwise equality for all $x \in \mathbb{R}^d$ (resp. for a.e. $x \in \mathbb{R}^d$) if and only if $M^{-1} M^{-\top} = \text{const}_{(\alpha, M, b)} I$ and $M^{-1} e_1 = \alpha^{p-1} \text{const}_{(\alpha, M, b)} e_1$. These last two equalities are equivalent to $\text{const}_{(\alpha, M, b)} = \alpha^{2-2p}$ and $M = \alpha^{p-1} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ with $Q \in \mathcal{O}(d-1)$. Finally note that for $u \in H^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ both sides of (10) belong to $L^2(\mathbb{R}^d)$ and since they are equal for a.e. $x \in \mathbb{R}^d$, the equality is an equality in L^2 . \square

Therefore, we consider the group $G := (\mathbb{R}_+ \times \text{SO}(d-1)) \ltimes \mathbb{R}^d$ whose elements we denote by $g = (\alpha, Q, b)$, with components $\alpha \in (0, \infty)$, $Q \in \text{SO}(d-1)$, $b \in \mathbb{R}^d$. The multiplication of two elements $(\alpha_1, Q_1, b_1), (\alpha_2, Q_2, b_2) \in G$ is given by

$$(\alpha_1, Q_1, b_1) \bullet (\alpha_2, Q_2, b_2) := \left(\alpha_1 \alpha_2, Q_1 Q_2, b_1 + \alpha_1 \begin{pmatrix} 1 & 0 \\ 0 & Q_1 \end{pmatrix} b_2 \right).$$

The group G is a non-compact, path connected Lie-group with unity element $\mathbb{1} = (1, I_{d-1}, 0)$ and the inverse of an element $(\alpha, Q, b) \in G$ is given by $(\alpha, Q, b)^{-1} = \left(\frac{1}{\alpha}, Q^\top, -\frac{1}{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & Q^\top \end{pmatrix} b \right)$. Moreover, its Lie-algebra (tangent space at $\mathbb{1}$) is

$$\mathfrak{g} = T_{\mathbb{1}} G = \mathbb{R} \times \mathfrak{so}(d-1) \times \mathbb{R}^d.$$

For later use we denote the left-multiplication by some element $g_0 = (\alpha_0, Q_0, b_0) \in G$, by

$$L_{g_0} : G \rightarrow G, \quad g \mapsto g_0 \bullet g,$$

and the derivative (tangential) of L_{g_0} at a point $g_1 = (\alpha_1, Q_1, b_1) \in G$ is

$$(11) \quad T_{g_1} L_{g_0} : T_{g_1} G \rightarrow T_{g_0 \bullet g_1} G, \quad \mu \mapsto T_{g_1} L_{g_0} \mu = \left(\alpha_0 \mu_1, Q_0 \mu_2, \alpha_0 \begin{pmatrix} 1 & 0 \\ 0 & Q_0 \end{pmatrix} \mu_3 \right).$$

From (11) we easily obtain the exponential map for G , i.e. the mapping $\exp : \mu \in \mathfrak{g} \rightarrow g(1) \in G$, where g is the solution to the differential equation $g'(t) = T_{\mathbb{1}} L_{g(t)} \mu$, $g(0) = \mathbb{1}$. For $\mu = (\mu_1, \mu_2, \mu_3) = (a, S, v) \in \mathfrak{g}$ we have the formula

$$(12) \quad \exp(\mu) = \left(\exp(a), \exp(S), \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \begin{pmatrix} a & 0 \\ 0 & a I_{d-1} + S \end{pmatrix}^k v \right),$$

where $\exp(a)$ is the standard exponential in \mathbb{R} and $\exp(S)$ is the standard matrix exponential. Note that by associativity $L_{g_0}^{-1} = L_{g_0^{-1}}$ and therefore we have the identity

$$(13) \quad (T_{g_0} L_{g_0^{-1}})^{-1} = T_{\mathbf{1}} L_{g_0}.$$

Remark 2.4. The Lie group G can be written as a matrix Lie group. More precisely,

$$(14) \quad \mathcal{M} : G \rightarrow \mathbb{R}^{d+1, d+1}, \quad \mathcal{M} : g = (\alpha, Q, b) \mapsto \mathcal{M}_g = \begin{pmatrix} \alpha & 0 & b_1 \\ 0 & \alpha Q & b_{2:d} \\ 0 & 0 & 1 \end{pmatrix},$$

where as before $\alpha > 0$, $Q \in \text{SO}(d-1)$, $b = (b_1, b_2, \dots, b_d)^\top \in \mathbb{R}^d$ and we abbreviate $b_{2:d} = (b_2, \dots, b_d)^\top$. For $g = (\alpha, Q, b), g' = (\alpha', Q', b') \in G$ holds

$$\mathcal{M}_g \mathcal{M}_{g'} = \begin{pmatrix} \alpha\alpha' & 0 & \alpha b'_1 + b_1 \\ 0 & \alpha\alpha' Q Q' & \alpha Q b'_{2:d} + b_{2:d} \\ 0 & 0 & 1 \end{pmatrix} = \mathcal{M}_{g \bullet g'},$$

and obviously $\mathcal{M}_{\mathbf{1}} = I_{d+1} \in \mathbb{R}^{d+1, d+1}$, so that \mathcal{M} is a group homomorphism from (G, \bullet) to the matrix Lie group (closed subgroup of $\text{GL}(d+1; \mathbb{R})$)

$$\mathcal{M}_G := \left\{ M = \begin{pmatrix} \alpha & 0 & b_1 \\ 0 & \alpha Q & b_{2:d} \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{d+1, d+1} : \alpha \in (0, \infty), \begin{pmatrix} b_1 \\ b_{2:d} \end{pmatrix} \in \mathbb{R}^d, Q \in \text{SO}(d-1) \right\}.$$

The corresponding matrix Lie algebra of \mathcal{M}_G is

$$(15) \quad \mathfrak{m}_G = T_{I_{d+1}} \mathcal{M}_G = \left\{ m = \begin{pmatrix} a & 0 & v_1 \\ 0 & aI_{d-1} + S & v_{2:d} \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{R}, \begin{pmatrix} v_1 \\ v_{2:d} \end{pmatrix} \in \mathbb{R}^d, S \in \mathfrak{so}(d-1) \right\}.$$

In this matrix setting the derivative of the left multiplication in \mathcal{M}_G by some element \mathcal{M}_{g_0} with $g_0 = (\alpha_0, Q_0, b_0) \in G$ of course is the left multiplication by a matrix:

$$L_{\mathcal{M}_{g_0}} : \mathcal{M}_G \rightarrow \mathcal{M}_G, \quad \mathcal{M}_g \mapsto L_{\mathcal{M}_{g_0}} \mathcal{M}_g = \mathcal{M}_{g_0} \mathcal{M}_g = \mathcal{M}_{g_0 \bullet g}.$$

The tangential of $L_{\mathcal{M}_{g_0}}$ at the identity is easily calculated to be

$$T_I L_{\mathcal{M}_{g_0}} : \mathfrak{m}_G \rightarrow \mathfrak{m}_G, \quad m = \begin{pmatrix} a & 0 & v_1 \\ 0 & aI + S & v_{2:d} \\ 0 & 0 & 0 \end{pmatrix} \mapsto \mathcal{M}_{g_0} m = \begin{pmatrix} \alpha a & 0 & \alpha v_1 \\ 0 & \alpha Q a + \alpha Q S & \alpha Q v_{2:d} \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, in this case the exponential map in \mathcal{M}_G is simply given by the matrix exponential, and one

finds for $m = \begin{pmatrix} a & 0 & v_1 \\ 0 & aI + S & v_{2:d} \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{m}_G$ the formula

$$(16) \quad \exp(m) = \begin{pmatrix} \exp(a) & 0 & \sum_{k=0}^{\infty} \frac{1}{(k+1)!} a^k v_1 \\ 0 & \exp(a) \exp(S) & \sum_{k=0}^{\infty} \frac{1}{k+1} (aI + S)^k v_{2:d} \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that comparing the entries in (16) we again obtain formula (12) for the exponential map in G .

3. THE ACTION AND ITS GENERATORS

In the previous section we derived a continuous family of symmetries (Lie group) which basically commutes with the nonlinear vector field F from (8), see (10). In this section we now consider the analytic properties of how these symmetries act on functions. We also calculate the generators of these actions. From now on let the parameter $p > 1$ in the vector field $F(u) = \nu \Delta u + \frac{1}{p} \frac{\partial}{\partial x_1}(|u|^p)$ be fixed. We denote by a the action of the symmetry group G on functions $v : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$(17) \quad a(\alpha, Q, b)v(x) := a((\alpha, Q, b), v)(x) := \alpha^{-1}v\left(\alpha^{1-p}\tilde{Q}^\top(x-b)\right) \quad \forall x \in \mathbb{R}^d,$$

where \tilde{Q} denotes the augmented matrix $\tilde{Q} = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ for brevity. Obviously, a is a linear left action. Furthermore, we denote by m the group homomorphism

$$(18) \quad m : G \rightarrow (\mathbb{R}_+, \cdot), \quad m(\alpha, Q, b) = \alpha^{2-2p}.$$

First we show that the group action a of G is indeed a strongly continuous group action on $\mathcal{C}_0^k(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d)$.

Lemma 3.1. *Let $X = \mathcal{C}_0^k(\mathbb{R}^d)$ or $X = H^k(\mathbb{R}^d)$. Let $p > 1$ be fixed and the action a of G on functions be given by (17). Then $a : G \times X \rightarrow X, (g, v) \mapsto a(g)v$ is a strongly continuous group action, i.e.*

- (i) $a(g) \in \text{GL}(X) \quad \forall g \in G,$
- (ii) $a(\mathbb{1}) = \text{id}_X,$
- (iii) $a(g \bullet h) = a(g) \circ a(h) \quad \forall g, h \in G,$
- (iv) $\lim_{g \rightarrow \mathbb{1}} a(g)v = v \quad \forall v \in X.$

Using the Banach-Steinhaus Theorem, Lemma 3.1 easily implies that a is continuous:

Corollary 3.2. *The mapping $a : G \times X \rightarrow X, (g, v) \mapsto a(g)v$ is continuous.*

Proof of Lemma 3.1. Throughout the proof we always use $g = (\alpha, Q, b) \in G$. We begin with some preliminaries. For $v \in \mathcal{C}_0^0(\mathbb{R}^d)$ holds

$$(19) \quad \|a(g)v\|_\infty = \frac{1}{\alpha} \|v\|_\infty,$$

and for $v \in \mathcal{S}(\mathbb{R}^d)$ we obtain by using the transformation formula the equality

$$(20) \quad \|a(g)v\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{1}{\alpha^2} \left| v\left(\frac{1}{\alpha^p}\tilde{Q}^\top(x-b)\right) \right|^2 dx = \alpha^{pd-2} \|v\|_{L^2}^2,$$

which extends to all of $L^2(\mathbb{R}^d)$ by density of $\mathcal{S}(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$.

Now let v be sufficiently smooth, e.g. $v \in \mathcal{S}(\mathbb{R}^d)$, which is dense in both $\mathcal{C}_0^k(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d)$ (e.g. [16, § 2.6]). For $i \in 1, \dots, d$ then follow with the chain rule

$$\frac{\partial}{\partial x_i} [a(\alpha, Q, b)v](x) = \frac{1}{\alpha} Dv \left(\frac{1}{\alpha^p} \tilde{Q}^\top(x-b) \right) \frac{1}{\alpha^p} \tilde{Q}^\top e_i = \frac{1}{\alpha^p} [a(g)v_i](x),$$

where we denote $v_i(x) = Dv(x)\tilde{Q}^\top e_i$, so that inductively for every multi-index $\beta \in \mathbb{N}^d$ holds

$$(21) \quad D^\beta [a(\alpha, Q, b)v] = \frac{1}{\alpha^{p|\beta|}} \left[a(g) \left(D^{|\beta|} v(\cdot) [\tilde{Q}^\top e^\beta] \right) \right].$$

Here we abbreviate $[\tilde{Q}^\top e^\beta] = [\tilde{Q}^\top e_1, \dots, \tilde{Q}^\top e_1, \dots, \tilde{Q}^\top e_d]$, where each $\tilde{Q}^\top e_i$ is repeated β_i -times, $|\beta| = \sum_{i=1}^d \beta_i$ as usual for multi-indices, and $D^{|\beta|}v$ is the β 'th total derivative of v . Because \tilde{Q} is an orthogonal matrix and $|e_i| = 1$ for all $i = 1, \dots, d$, the absolute value of the function $D^{|\beta|}v(\cdot)(\tilde{Q}^\top e^\beta)$ is pointwise bounded by

$$(22) \quad \left| D^{|\beta|}v(x)(\tilde{Q}^\top e^\beta) \right| \leq \text{const}(|\beta|) \sum_{\nu \leq |\beta|} |D^\nu v(x)|,$$

where $\text{const}(|\beta|)$ only depends on $|\beta|$. Now we are ready to prove (i). Linearity and invertibility are obvious so that it remains to show boundedness in the respective spaces. First let $v \in \mathcal{C}_0^k(\mathbb{R}^d)$. Then for $g = (\alpha, Q, b) \in G$ follows

$$\begin{aligned} \|a(g)v\|_{\mathcal{C}^k} &= \max_{|\beta| \leq k} \frac{1}{\alpha^{p|\beta|}} \|a(g)(D^{|\beta|}v(\cdot)[\tilde{Q}^\top e^\beta])\|_\infty \\ &\leq \left(1 + \frac{1}{\alpha^{pk}}\right) \max_{|\beta| \leq k} \sup_x \left| \frac{1}{\alpha} D^{|\beta|}v\left(\frac{1}{\alpha^p} \tilde{Q}^\top(x-b)\right) [\tilde{Q}^\top e^\beta] \right| \\ &\leq \left(1 + \frac{1}{\alpha^{pk}}\right) \frac{1}{\alpha} \text{const}(k) \|v\|_{\mathcal{C}^k}, \end{aligned}$$

where we used (21), (22), and (19). This implies

$$(23) \quad \|a(g)\|_{\text{GL}(\mathcal{C}_0^k)} \leq \text{const}(\alpha, k),$$

where $\text{const}(\alpha, k)$ can be chosen independently of α for all α from a fixed compact subset of $(0, \infty)$ and hence for all g from a fixed compact subset of G .

Now consider the boundedness in $H^k(\mathbb{R}^d)$. Let $v \in \mathcal{S}(\mathbb{R}^d)$ and recall that the Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ (e.g. [21, Thm. 7.4]). We first observe

$$(24) \quad (a(g)v)^\wedge(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix^\top \xi} \frac{1}{\alpha} v\left(\frac{1}{\alpha^p} \tilde{Q}^\top(x-b)\right) dx = e^{-ib^\top \xi} \alpha^{dp-1} \widehat{v}(\alpha^p \tilde{Q}^\top \xi),$$

so that we find for the $\|\cdot\|_{\dot{H}^k}$ -norm (see (4))

$$\begin{aligned} \|a(g)v\|_{\dot{H}^k} &= \alpha^{2(dp-1)} \int_{\mathbb{R}^d} \langle \xi \rangle^{2k} |\widehat{v}(\alpha^p \tilde{Q}^\top \xi)|^2 d\xi = \alpha^{dp-2} \int_{\mathbb{R}^d} \langle \tilde{Q} \alpha^{-p} z \rangle^{2k} |\widehat{v}(z)|^2 dz \\ &\leq \alpha^{dp-2} (1 + \alpha^{-2pk}) \|v\|_{\dot{H}^k}^2 = \text{const}(\alpha, d, k) \|v\|_{\dot{H}^k}^2. \end{aligned}$$

Again the constant $\text{const}(\alpha, d, k)$ is uniformly bounded in compact subsets of G . Because $\mathcal{S}(\mathbb{R}^d)$ is dense in $H^k(\mathbb{R}^d)$ and the norm $\|\cdot\|_{\dot{H}^k}$ is equivalent to the $\|\cdot\|_{H^k}$ -norm, the same estimate holds for any $v \in H^k$ and

$$(25) \quad \|a(g)\|_{\text{GL}(H^k)} \leq \text{const}(\alpha, k, d)$$

follows. This finishes the proof of (i).

By definition of the action a also (ii) and (iii) hold.

For the proof of (iv) we again first consider the \mathcal{C}_0^k -case:

Let $v \in \mathcal{C}_0^0$. For $g = (\alpha, Q, b) \in G$ we then obtain the estimate

$$\begin{aligned} (26) \quad \|a(g)v - v\|_\infty &= \sup_{x \in \mathbb{R}^d} \left| \alpha^{-1} v(\alpha^{-p} \tilde{Q}^\top(x-b)) - v(x) \right| \\ &\leq \alpha^{-1} \sup_{|x| \geq R} \left| v(\alpha^{-p} \tilde{Q}^\top(x-b)) \right| + \sup_{|x| \geq R} |v(x)| \\ &\quad + \alpha^{-1} \sup_{|x| \leq R} \left| v(\alpha^{-p} \tilde{Q}^\top(x-b)) - v(x) \right| + |\alpha^{-1} - 1| \sup_{|x| \leq R} |v(x)|. \end{aligned}$$

To show that $\|a(g)v - v\|_\infty \rightarrow 0$ as $g \rightarrow \mathbb{1}$, let $\epsilon > 0$ be given. Since $\lim_{|x| \rightarrow \infty} |v(x)| = 0$, there is $R > 0$, so that the first two summands on the right hand side of (26) are smaller than ϵ for all $|b| < 1$ and $|1 - \alpha| < \frac{1}{2}$. Because v is uniformly continuous on compact subsets and $\alpha^{-p} \tilde{Q}^\top(x-b) \rightarrow x$ uniformly on $\{|x| \leq R\}$ as $\alpha \rightarrow 1, Q \rightarrow I, b \rightarrow 0$, the third summand converges to zero as $g \rightarrow \mathbb{1}$. Also the last summand converges to zero as $g \rightarrow \mathbb{1}$, since $\alpha \rightarrow 1$ and v is uniformly bounded.

Now let $v \in \mathcal{C}_0^k$ and $\beta \in \mathbb{N}^d$ be some multi-index with $0 < |\beta| \leq k$. Then we use (21) to estimate for $g = (\alpha, Q, b) \in G$

$$(27) \quad \begin{aligned} \|D^\beta([a(g)v] - v)\|_\infty &\leq \left|1 - \alpha^{-|\beta|}\right| \|a(g)\|_{\text{GL}(\mathcal{C}_0^0)} \|D^{|\beta|}v(\cdot)[\tilde{Q}^\top e^\beta]\|_\infty \\ &\quad + \|a(g)\|_{\text{GL}(\mathcal{C}_0^0)} \|D^{|\beta|}v(\cdot)[\tilde{Q}^\top e^\beta] - D^{|\beta|}v(\cdot)[Ie^\beta]\|_\infty \\ &\quad + \|a(g)D^\beta v(\cdot) - D^\beta v(\cdot)\|_\infty. \end{aligned}$$

The first summand on the right hand side of (27) converges to zero as $g \rightarrow \mathbb{1}$, because $\|a(g)\|_{\text{GL}(\mathcal{C}_0^0)}$ is uniformly bounded for g from a compact subset of G by (23) and the factor $\|D^{|\beta|}v(\cdot)[\tilde{Q}^\top e^\beta]\|_\infty$ is uniformly bounded by (22). The second summand converges to zero as $g \rightarrow \mathbb{1}$ again because of (23) and since $[\tilde{Q}^\top e^\beta] \rightarrow [Ie^\beta]$ in $(\mathbb{R}^d)^{|\beta|}$ (recall the abbreviation introduced above) and the $|\beta|$ -multilinear forms $D^{|\beta|}v(x)$ are bounded independently of x . Finally, also the third summand converges to zero as $g \rightarrow \mathbb{1}$ because of the \mathcal{C}_0^0 -case, considered above. This proves (iv) for the $\mathcal{C}_0^k(\mathbb{R}^d)$ -case.

For the H^k -case it suffices to consider $v \in \mathcal{S}(\mathbb{R}^d)$ because of estimate (25) and the density of $\mathcal{S}(\mathbb{R}^d)$ in $H^k(\mathbb{R}^d)$. Therefore, let $v \in \mathcal{S}(\mathbb{R}^d)$ and let $(g_n)_{n \in \mathbb{N}} \subset G$ be a sequence with $g_n \rightarrow \mathbb{1}$ as $n \rightarrow \infty$. Using (24) we find

$$\|a(g_n)v - v\|_{H^k}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^{2k} \left| \alpha_n^{dp-1} \hat{v}(\alpha_n^p \tilde{Q}_n^\top \xi) e^{-ib_n^\top \xi} - \hat{v}(\xi) \right|^2 d\xi.$$

The integrand converges pointwise to zero as $n \rightarrow \infty$ and is bounded by

$$(28) \quad 2\langle \xi \rangle^{2k} \alpha_n^{2dp-2} \hat{v}(\alpha_n^p \tilde{Q}_n^\top \xi)^2 + 2\langle \xi \rangle^{2k} \hat{v}(\xi)^2.$$

For n_0 with $|\alpha_n - 1| \leq \frac{1}{2}$ for all $n \geq n_0$, both summands of (28) are uniformly integrable independently of $n \geq n_0$. Finally, for any given $\epsilon > 0$ there exists $R > 0$, such that for all $n \geq n_0$ holds

$$\int_{|x| > R} 2\langle \xi \rangle^{2k} \left(\alpha_n^{2dp-2} \hat{v}(\alpha_n^p \tilde{Q}_n^\top \xi)^2 + \hat{v}(\xi)^2 \right) d\xi \leq \epsilon.$$

Thus, the Vitali convergence theorem (e.g. [8, VI.5.6]) applies and yields

$$\lim_{n \rightarrow \infty} \|a(g_n)v - v\|_{H^k} = 0.$$

□

With the above definitions of G , a and m , and because of Lemma 3.1, we can rephrase Proposition 2.3 as follows:

Proposition 3.3. *For all $v \in \mathcal{C}^2(\mathbb{R}^d)$ (resp. $v \in H^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$) hold for all $g \in G$*

$$(29) \quad F(a(g)v) = m(g)a(g)F(v),$$

as identities in \mathcal{C}^0 (resp. in L^2).

As before, let $X = \mathcal{C}_0^k(\mathbb{R}^d)$ or $X = H^k(\mathbb{R}^d)$. For fixed $v \in X$ we denote by $T_g a v : T_g G \rightarrow T_{a(g)v} X$ the tangential of the mapping $G \ni g \mapsto a(g)v \in X$ at $g \in G$, if it exists. The evaluation of $T_g a v$ at $\mu \in T_g G$ is denoted by $T_g a v[\mu]$. To motivate that the set of all $v \in X$ for which the tangential exists is actually a useful set, note that by Lemma 3.1 and the properties of the exponential map for each $\mu \in \mathfrak{g}$ the mapping $S_\mu : t \mapsto a(\exp(\mu t))$ defines a strongly continuous semigroup on X . Hence the generator is a densely defined and closed operator on X . The generator coincides with the directional derivative of the group action at $\mathbb{1}$ in the direction μ . This motivates to consider for a basis $\epsilon_1, \dots, \epsilon_{\dim(\mathfrak{g})}$ of \mathfrak{g} the subset Y_0 of X defined as

$$(30) \quad Y_0 := \left\{ v \in X : \exists T_{\mathbb{1}} a v[\epsilon_j] = \lim_{t \rightarrow 0} \frac{1}{t} (a(\exp(\epsilon_j t))v - v), j = 1, \dots, \dim(\mathfrak{g}) \right\}.$$

Using Lemma 3.1 and the results of [22, § 4], we obtain that Y_0 indeed is a dense subset of X , and Y_0 becomes a Banach-space, when endowed with the norm

$$\|v\|_{Y_0} = \|v\|_X + \sup_{j=1, \dots, \dim \mathfrak{g}} \|T_1 a v[\epsilon_j]\|_X.$$

Moreover, G acts continuously on $(Y_0, \|\cdot\|_{Y_0})$ and for fixed $v \in Y_0$ the mapping

$$G \ni g \mapsto a(g)v \in X$$

is continuously differentiable. We summarize the above discussion in the following lemma.

Lemma 3.4. *Let $X = \mathcal{C}_0^k(\mathbb{R}^d)$ or $X = H^k(\mathbb{R}^d)$ and let Y_0 be defined by (30). Then $Y_0 \subset X$ is a dense subset and the action a , defined in (17) has the following properties:*

- (a) *For fixed $g \in G$, the operator $a(g)$ is a bounded linear operator on X and on Y_0 .*
- (b) *For fixed $v \in X$, resp. $v \in Y_0$, the mapping $G \ni g \mapsto a(g)v$ is continuous into X , resp. Y_0 .*
- (c) *For fixed $v \in Y_0$, the mapping $G \ni g \mapsto a(g)v \in X$, is continuously differentiable.*

Example 3.5. For later use, we now explicitly calculate for $d = 1$, $d = 2$, and $d = 3$ spatial dimensions for specific choices of bases $\epsilon_1, \dots, \epsilon_{\dim(\mathfrak{g})}$ of \mathfrak{g} the generators $v \mapsto T_1 a v[\epsilon_j]$ of the semigroups $(S_{\epsilon_j}(t))_{t \geq 0}$ and the space Y_0 . As before, we may restrict the calculation to $v \in \mathcal{S}(\mathbb{R}^d)$, which is dense in both, $\mathcal{C}_0^k(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d)$.

- (i) For $d = 1$ the symmetry group is $G = (\mathbb{R}_+ \times \emptyset) \ltimes \mathbb{R} \hat{=} \mathbb{R}_+ \ltimes \mathbb{R}$. As a basis $\{\epsilon_1, \epsilon_2\}$ of $\mathfrak{g} = T_1 G = \mathbb{R} \times \mathbb{R}$ we choose

$$\epsilon_1 = (1, 0), \epsilon_2 = (0, 1).$$

From formula (12) for the exponential map we obtain $\exp(\epsilon_1 t) = (e^t, 0)$ and $\exp(\epsilon_2 t) = (0, t)$, so that together with (17) follow

$$\begin{aligned} T_1 a v[\epsilon_1] &= \lim_{t \searrow 0} \frac{e^{-t} v(e^{(1-p)t} \cdot) - v(\cdot)}{t} = -v + (1-p)xv_x, \\ T_1 a v[\epsilon_2] &= \lim_{t \searrow 0} \frac{v(\cdot - t) - v(\cdot)}{t} = -v_x. \end{aligned}$$

Therefore, $Y_0 = \{v \in X : v_x \in X, xv_x \in X\}$.

- (ii) For $d = 2$ we obtain $G = (\mathbb{R}_+ \times \{1\}) \ltimes \mathbb{R}^2$. As basis of $\mathfrak{g} = T_1 G = \mathbb{R} \times \{0\} \times \mathbb{R}^2$ we choose

$$\epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 0, e_1), \epsilon_3 = (0, 0, e_2).$$

Similar considerations as in the 1-dimensional case lead to

$$\begin{aligned} T_1 a v[\epsilon_1] &= -v + (1-p)x^\top \nabla v, \\ T_1 a v[\epsilon_2] &= -v_x, \\ T_1 a v[\epsilon_3] &= -v_y, \end{aligned}$$

and $Y_0 = \{v \in X : v_x, v_y, xv_x + yv_y \in X\}$.

- (iii) For $d = 3$ we have $G = (\mathbb{R}_+ \times \text{SO}(2)) \ltimes \mathbb{R}^3$ and choose the basis

$$\epsilon_1 = (1, 0, 0), \epsilon_2 = \left(0, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0\right), \epsilon_j = (0, 0, e_{j-2}), j = 3, 4, 5$$

of $\mathfrak{g} = T_1 G = \mathbb{R} \times \text{span}\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} \times \mathbb{R}^3$. Analogous calculations as in the $d = 1$ and $d = 2$ case now yield the generators

$$\begin{aligned} T_1 a v[\epsilon_1] &= -v + (1-p)x^\top \nabla v, \\ T_1 a v[\epsilon_2] &= x^\top \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \nabla v, \end{aligned}$$

$$T_1 a v[\epsilon_j] = -v_{x_{j-2}}, \quad j = 3, 4, 5$$

and $Y_0 = \{v \in X : v_x, v_y, v_z, xv_x + yv_y + zv_z, zv_y - yv_z \in X\}$, where we interchangeably write $v_x = v_{x_1}$, $v_y = v_{x_2}$, and $v_z = v_{x_3}$.

In the following we will often use the spaces $\mathcal{C}_{\text{scal}}^2(\mathbb{R}^d)$ and $H_{\text{scal}}^2(\mathbb{R}^d)$, which are defined as

$$(31) \quad \mathcal{C}_{\text{scal}}^2(\mathbb{R}^d) := \left\{ v \in \mathcal{C}_0^2(\mathbb{R}^d) : T_1 a v[\mu] \in \mathcal{C}_0^0(\mathbb{R}^d) \quad \forall \mu \in \mathfrak{g} \right\}$$

and

$$(32) \quad H_{\text{scal}}^2(\mathbb{R}^d) := \left\{ v \in H^2(\mathbb{R}^d) : T_1 a v[\mu] \in L^2(\mathbb{R}^d) \quad \forall \mu \in \mathfrak{g} \right\}.$$

Endowed with a suitable norm, these become Banach-spaces (respectively Hilbert-spaces). For concreteness we choose in the case of one spatial dimension the norms $\|v\|_{\mathcal{C}_{\text{scal}}^2} = \|v\|_{\mathcal{C}^2} + \|xv_x\|_{\infty}$, respectively $\|v\|_{H_{\text{scal}}^2}^2 = \|v\|_{H^2}^2 + \|xv_x\|_{L^2}^2$. In the $d = 2$ case the norms $\|v\|_{\mathcal{C}_{\text{scal}}^2} = \|v\|_{\mathcal{C}^2} + \|x^\top \nabla v\|_{\infty}$, respectively $\|v\|_{H_{\text{scal}}^2}^2 = \|v\|_{H^2}^2 + \|x^\top \nabla v\|_{L^2}^2$, are suitable. And in the $d = 3$ case, we choose $\|v\|_{\mathcal{C}_{\text{scal}}^2} = \|v\|_{\mathcal{C}^2} + \|x^\top \nabla v\|_{\infty} + \|-yv_z + zv_y\|_{\infty}$, respectively $\|v\|_{H_{\text{scal}}^2}^2 = \|v\|_{H^2}^2 + \|x^\top \nabla v\|_{L^2}^2 + \|-yv_z + zv_y\|_{L^2}^2$.

Remark 3.6. It is easy to see that the generator of the scaling action, given by

$$T_1 a v[(1, 0, 0)] = -v + (1 - p)x^\top \nabla v = -v + (1 - p)\text{div}(xv) + d(p - 1)v$$

in any space dimension, is in divergence form for all $v \in \mathcal{S}(\mathbb{R}^d)$ if and only if

$$1 = d(p - 1) \Leftrightarrow p = \frac{d+1}{d}.$$

In contrast to this, the generators of the translation and rotation actions are always in divergence form. This is precisely the case for the action of the symmetry belonging to, what we called, the “conservative Burgers’ equation” in the Section 1. This observation is precisely the reason, why we called the case $p = \frac{d+1}{d}$ in (2) the conservative Burgers’ equation in the first place. Note that the “conservative Burgers’ equation” coincides with the “multi-dimensional Burgers’ equation” only in the case $d = 1$.

4. FREEZING SIMILARITY SOLUTIONS

In this section we derive the equations for the numerical method of freezing similarity solutions in generalized Burgers’ equation. In the first part we derive the equation (2) in new, time-dependent, coordinates, based on the symmetries obtained in Sections 2 and 3. In the second part we augment this new system with algebraic constraints to deal with the arbitrariness introduced by the new coordinates. In the following we denote $X = \mathcal{C}_0^0(\mathbb{R}^d)$ and then $Y_1 = \mathcal{C}_{\text{scal}}^2(\mathbb{R}^d)$ or $X = L^2(\mathbb{R}^d)$ and then $Y_1 = H_{\text{scal}}^2(\mathbb{R}^d)$, respectively.

4.1. Co-moving Frames and Similarity Solutions. The basic idea of the method is to split the time evolution of the solution of the Cauchy problem for (8) into a part that treats the evolution which mainly takes place in the group orbit of the profile and a part that takes the evolution of the profile in the remaining directions into account. Formally, this is done by making the ansatz

$$(33) \quad u(t) = a(g(\sigma(t)))v(\sigma(t)),$$

where a is the left action of the symmetry group G , defined in (17), g is a smooth curve in the group G , σ is an orientation preserving diffeomorphism of two intervals, describing a scaling of time, and v is a time dependent profile, taking values in Y_1 .

A solution of the Cauchy-problem, whose evolution solely takes place in the group orbit, we call a similarity solution. This is made precise in the next definition.

Definition 4.1 (Similarity solution). *We call u a similarity solution of $u_t = F(u)$ with profile $\underline{v} \in Y_1$, if there exists an open interval $J \subset \mathbb{R}$, a differentiable map $\sigma : J \rightarrow \mathbb{R}$ with $\dot{\sigma}(t) > 0$ for all $t \in J$, and a differentiable curve $g \in \mathcal{C}^1(\sigma(J), G)$, such that $u(t) = a(g(\sigma(t)))\underline{v}$ is a solution of $u_t = F(u)$.*

The key for the freezing method is now the following Theorem 4.2, which relates the differential equations satisfied by the two functions u and v from the ansatz (33).

Theorem 4.2. *Let $u_0 \in Y_1$ and $\mu \in \mathcal{C}([0, \hat{T}]; \mathfrak{g})$, $\hat{T} > 0$ be given. Then there exist unique maximally extended solutions $g \in \mathcal{C}^1([0, \hat{T}]; G)$ of*

$$(34a) \quad g'(\tau) = T_{\mathbb{1}} L_{g(\tau)} \mu(\tau), \quad g(0) = \mathbb{1},$$

and $\sigma \in \mathcal{C}^1([0, T]; [0, \hat{T}])$ of

$$(34b) \quad \dot{\sigma}(t) = m(g(\sigma(t))), \quad \sigma(0) = 0.$$

The function $\sigma : [0, T] \rightarrow [0, \hat{T})$ is a diffeomorphism. Furthermore, the following statements hold true:

- (i) *If $u \in \mathcal{C}([0, T]; Y_1) \cap \mathcal{C}^1([0, T]; X)$ solves the Cauchy problem for (8) with initial condition $u(0) = u_0 \in Y_1$, then $v : \tau \mapsto a(g(\tau)^{-1})u(\sigma^{-1}(\tau))$ belongs to $\mathcal{C}([0, T]; Y_1) \cap \mathcal{C}^1([0, T]; X)$ and solves*

$$(35) \quad v_\tau = F(v) - T_{\mathbb{1}} a v[\mu], \quad v(0) = u_0.$$

- (ii) *If $v \in \mathcal{C}([0, \hat{T}]; Y_1) \cap \mathcal{C}^1([0, \hat{T}]; X)$ solves the Cauchy problem (35), then $u : t \mapsto a(g(\sigma(t)))v(\sigma(t))$ belongs to $\mathcal{C}([0, T]; Y_1) \cap \mathcal{C}^1([0, T]; X)$ and solves the Cauchy problem for (8) with $u(0) = u_0$.*

Proof. By (11), the differential equation for the first component of g in (34a) decouples and can be solved first. It follows that the solution for the first component exists globally in $[0, \hat{T})$. In a subsequent step, the remaining differential equations for the other components of g can be solved. More precisely, knowing the first component of g in $[0, \hat{T})$, the remaining ODEs become linear and hence the solution g exists globally and belongs to $\mathcal{C}^1([0, \hat{T}]; G)$. In the next step, one uses that g is known and $m \circ g \in \mathcal{C}^1([0, \hat{T}], \mathbb{R}_+)$. Therefore, (34b) has a unique maximally extended solution $\sigma \in \mathcal{C}^1([0, T]; [0, \hat{T}))$. Because of the differential equation $\dot{\sigma}(t) = m(g(\sigma(t))) > 0$ for all $t \in [0, T)$ and $\sigma : [0, T) \rightarrow [0, \hat{T})$ is a diffeomorphism.

Proof of (i). The smoothness of v follows from Lemma 3.4 and the assumptions on u . Because a is a group homomorphism, $v(\tau) = a(g(\tau)^{-1})u(\sigma^{-1}(\tau))$ is equivalent to

$$(36) \quad a(g(\tau))v(\tau) = u(\sigma^{-1}(\tau)) \quad \forall \tau \in [0, \hat{T}).$$

First consider the left hand side of (36). For $h \in \mathbb{R}$, h small, with $\tau + h \in [0, \hat{T})$ we find as equalities in X

$$\begin{aligned} & a(g(\tau + h))v(\tau + h) - a(g(\tau))v(\tau) \\ &= a(g(\tau + h))(v(\tau + h) - v(\tau)) + a(g(\tau + h))v(\tau) - a(g(\tau))v(\tau) \\ &= a(g(\tau + h))(v_\tau(\tau)h + o(|h|)) + a(g(\tau))\left(a(L_{g(\tau)^{-1}}g(\tau + h))v(\tau) - a(L_{g(\tau)^{-1}}g(\tau))v(\tau)\right) \\ &= a(g(\tau + h))v_\tau(\tau)h + o(|h|) + a(g(\tau))T_{\mathbb{1}}a v(\tau)[T_{g(\tau)}L_{g(\tau)^{-1}}(g'(\tau)h + o(|h|))]. \end{aligned}$$

For the last equality one must use a chart of G at $g(\tau)$ and a chart of G at $\mathbb{1}$. Note that the first o -term in the last line belongs to X and the other one to $T_{g(\tau)}G$. The above equalities show

$$(37) \quad \lim_{h \rightarrow 0} \frac{1}{h} \left(a(g(\tau + h))v(\tau + h) - a(g(\tau))v(\tau) \right) = a(g(\tau))v_\tau(\tau) + a(g(\tau))T_{\mathbb{1}}a v(\tau)[T_{g(\tau)}L_{g(\tau)^{-1}}g'(\tau)],$$

where the limit exists in X . This is the derivative of the left hand side of (36) with respect to τ .

Differentiation of the right hand side of (36) at τ yields by the chain rule the identities

$$(38) \quad u_t(\sigma^{-1}(\tau)) \frac{1}{\dot{\sigma}(\sigma^{-1}(\tau))} = \frac{1}{m(g(\tau))} F(u(\sigma^{-1}(\tau))) = m(g(\tau)^{-1}) F(u(\sigma^{-1}(\tau))),$$

where we used (34b), (8), and that m is a group homomorphism.

Equating (37) and (38) and application of $a(g(\tau)^{-1})$ to both sides leads to

$$v_\tau(\tau) + T_1 a v(\tau) [T_{g(\tau)} L_{g(\tau)^{-1}} g'(\tau)] = a(g(\tau)^{-1}) m(g(\tau)^{-1}) F(u(\sigma^{-1}(\tau)))$$

as an equality in X . Using equation (34a) and the symmetry property, Lemma 3.3, then proves the asserted equality (35).

Proof of (ii). For all $t \in [0, T)$ let u be given by

$$(39) \quad u(t) = a(g(\sigma(t))) v(\sigma(t)).$$

The asserted smoothness of u follows from Lemma 3.4 and we may differentiate (39) with respect to t as in the proof of (i), which leads to

$$(40) \quad \begin{aligned} u_t(t) &= a(g(\sigma(t))) T_1 a v(\sigma(t)) [T_{g(\sigma(t))} L_{g(\sigma(t))^{-1}} g'(\sigma(t)) \dot{\sigma}(t)] + a(g(\sigma(t))) v_\tau(\sigma(t)) \dot{\sigma}(t) \\ &= a(g(\sigma(t))) (T_1 a v(\sigma(t)) [\mu(\sigma(t))] + v_\tau(\sigma(t))) m(g(\sigma(t))) \\ &= a(g(\sigma(t))) F(v(\sigma(t))) m(g(\sigma(t))) = F(u(t)), \end{aligned}$$

where we used the differential equations for g and σ in the second equality, the PDE for v in the third equality, and the symmetry property of F (see Proposition 3.3) in the last equality. Equation (40) holds as an equality in X for all $t \in [0, T)$, what finishes the proof. \square

Assume that u is a similarity solution in the sense of Definition 4.1, i.e.

$$(41) \quad u(t) = a(g(\sigma(t))) \underline{v}$$

for suitable functions $\sigma \in C^1(J, \mathbb{R})$, $g \in C^1(\sigma(J), G)$ and $\underline{v} \in Y_1$. By writing the right hand side of (41) in the form $a(g(\sigma(t))g(\sigma(0))^{-1})a(g(\sigma(0)))\underline{v}$, we may assume without loss of generality that $u_0 := u(0) = \underline{v}$. When we differentiate (41) and use that u solves the PDE (8), we obtain

$$u_t = F(a(g(\sigma(t)))\underline{v}) = a(g(\sigma(t))) T_1 a \underline{v} [T_{g(\sigma(t))} L_{g(\sigma(t))^{-1}} g'(\sigma(t)) \dot{\sigma}(t)].$$

With the symmetry property of F from Proposition 3.3, this equality is equivalent to

$$0 = F(\underline{v}) - T_1 a \underline{v} \left[T_{g(\sigma(t))} L_{g(\sigma(t))^{-1}} g'(\sigma(t)) \frac{\dot{\sigma}(t)}{m(g(\sigma(t)))} \right].$$

Under the assumption that $T_1 a \underline{v}[\epsilon_j]$, $j = 1, \dots, \dim(G)$ are linearly independent for any basis $\{\epsilon_1, \dots, \epsilon_{\dim(G)}\}$ of \mathfrak{g} , the function $\underline{\mu} := T_{g(\sigma(t))} L_{g(\sigma(t))^{-1}} g'(\sigma(t)) \frac{\dot{\sigma}(t)}{m(g(\sigma(t)))}$ must be independent of t . Therefore, the tuple $(\underline{v}, \underline{\mu}) \in Y_1 \times \mathfrak{g}$ solves

$$0 = F(\underline{v}) - T_1 a \underline{v}[\underline{\mu}], \text{ and } \underline{v} = u_0.$$

By Theorem 4.2 (ii) the function \underline{u} , given by $\underline{u}(t) = a(g(\underline{\sigma}(t)))\underline{v}$, where $\underline{g}(\tau) = \exp(\underline{\mu}\tau)$ for all $\tau \geq 0$ and $\underline{\sigma}$ solves $\dot{\underline{\sigma}}(t) = m(\underline{g}(\underline{\sigma}(t)))$, $\underline{\sigma}(0) = 0$, is a solution to the Cauchy-problem

$$u_t = F(u), \quad u(0) = u_0.$$

By uniqueness of this solution, we find $u(t) = a(g(\sigma(t)))\underline{v} = a(\exp(\underline{\mu}\underline{\sigma}(t)))\underline{v}$. Recollecting this discussion, we obtain the following result.

Proposition 4.3. *A function u is a similarity solution of $u_t = F(u)$ with profile $\underline{v} = u(0) \in Y_1$ for which $T_1 a \underline{v}$ is injective, if and only if there is $\underline{\mu} = (\underline{\mu}_1, \underline{\mu}_2, \underline{\mu}_3) \in \mathfrak{g} = \mathbb{R} \times \mathfrak{so}(d-1) \times \mathbb{R}^d$ with*

$$0 = F(\underline{v}) - T_1 a \underline{v}[\underline{\mu}].$$

Moreover, in this case $u(t) = a(\exp(\underline{\mu}\underline{\sigma}(t)))\underline{v}$, where $\underline{\sigma}$ is given by

$$\underline{\sigma}(t) = \frac{\ln((2p-2)\underline{\mu}_1 t + 1)}{(2p-2)\underline{\mu}_1},$$

which solves $\dot{\sigma} = m(\exp(\underline{\mu}\sigma)) = e^{(2-2p)\underline{\mu}_1\sigma}$, $\sigma(0) = 0$.

Remark 4.4. Note that if $p \neq \frac{d+1}{d}$ there exist no similarity solutions with a localized finite mass profile \underline{v} and nontrivial scaling $\underline{\mu}_1$. That is, \underline{v} and $\underline{\mu}_1$ satisfy $\underline{v} \in H^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $|\nabla \underline{v}(x)| + |\underline{v}(x)|^p \leq \text{const}|x|^d$, $\int_{\mathbb{R}^d} \underline{v}(x) dx \neq 0$, and $\underline{\mu}_1 \neq 0$.

This follows since by Proposition 4.3 also the function u , given by $u(t) = a(\exp(\underline{\mu}\underline{\sigma}(t)))\underline{v}$ has the decay properties of \underline{v} , so that Gauß' Theorem on the one hand shows

$$\frac{d}{dt} \int_{\mathbb{R}^d} u(x, t) dx = 0.$$

On the other hand, $p > 1$ and $\underline{\mu}_1 \neq 0$ imply $\underline{\alpha}(t) = ((2p-2)\underline{\mu}_1 t + 1)^{\frac{1}{2p-2}} \neq 1$ by Proposition 4.3, so that

$$\int_{\mathbb{R}^d} u(x, t) dx = \int_{\mathbb{R}^d} \alpha(t)^{-1} \underline{v}(\underline{\alpha}(t)^{1-p} \tilde{Q}(t)(x - b(t))) dx = \int_{\mathbb{R}^d} \alpha(t)^{dp-p-1} \underline{v}(y) dy.$$

Then $\int_{\mathbb{R}^d} u(x, t) dx$ is constant if and only if $dp - d = 1$.

Also, we can look at this from taking the point of view of the solution in the co-moving coordinates, i.e. v given by (33). Because of the divergence theorem and the assumption of localization (sufficiently fast decay of $|v(\xi)|$ and $|\nabla v(\xi)|$ as $|\xi| \rightarrow \infty$) this satisfies the identity

$$\frac{d}{d\tau} \int_{\mathbb{R}^d} v(\xi, \tau) d\xi = \int_{\mathbb{R}^d} (F(v) - T_1 a v[\underline{\mu}]) (\xi, \tau) d\xi = \mu_1(\tau)(1 + d - dp) \int_{\mathbb{R}^d} v(\xi, \tau) d\xi.$$

Therefore, the mass of v satisfies under the above assumptions of localization the equation

$$(42) \quad \int_{\mathbb{R}^d} v(\xi, \tau) d\xi = e^{(1+d-dp) \int_0^\tau \mu_0(\eta) d\eta} \int_{\mathbb{R}^d} v(\xi, 0) d\xi.$$

Remark 4.5. The results from Theorem 4.2 and Proposition 4.3 are not restricted to Burgers' type equations, but hold for all evolution equations which possess a similar symmetry structure.

Example 4.6. We now continue Example 3.5 and explicitly state the *co-moving equation* (35) and the *reconstruction equations* (34) for the cases of $d = 1, 2, 3$ spatial dimensions. This is needed for the actual implementation of the freezing method in the end.

To enhance readability of the equations, we as usual denote elements in G by $g = (\alpha, Q, b)$ with $\alpha \in \mathbb{R}_+$, $Q \in \text{SO}(d-1)$, $b \in \mathbb{R}^d$ and elements in $\mathfrak{g} = T_1 G$ are denoted by $\mu = (\mu_1, \mu_2, \mu_3)$, where $\mu_1 \in \mathbb{R}$, $\mu_2 = \sum_{j=1}^{\dim \mathfrak{so}(d-1)} \mu_2^j S_j \in \mathfrak{so}(d-1)$ with $S_1, \dots, S_{\dim \mathfrak{so}(d-1)}$ a basis of $\mathfrak{so}(d-1)$, μ_2 is identified with $(\mu_2^1, \dots, \mu_2^{\dim(\mathfrak{so}(d-1))})^\top \in \mathbb{R}^{\dim(\mathfrak{so}(d-1))}$, and $\mu_3 = \sum_{j=1}^d \mu_3^j e_j \in \mathbb{R}^d$. In particular, we have

$$(\mu_1, 0, 0) = \mu_1 \epsilon_1, \quad (0, \mu_2, 0) = \sum_{j=1}^{\dim \mathfrak{so}(d-1)} \mu_2^j \epsilon_{1+j}, \quad (0, 0, \mu_3) = \sum_{j=1}^d \mu_3^j \epsilon_{1+\dim \mathfrak{so}(d-1)+j},$$

where $\epsilon_1, \dots, \epsilon_{\dim \mathfrak{g}}$ is the canonical basis of \mathfrak{g} which was introduced in Example 3.5 in the cases $d = 1, 2, 3$.

- (i) For $d = 1$ we first assume that $\widehat{T} > 0$ and $\mu \in \mathcal{C}([0, \widehat{T}); \mathfrak{g})$ are given. The generators $T_1 a v[\epsilon_1]$ and $T_1 a v[\epsilon_2]$ are calculated in Example 3.5 (i) and inserting them into the *co-moving equation* (35) we obtain the explicit form

$$(43a) \quad v_\tau = \nu v_{xx} - \frac{1}{p}(|v|^p)_x + \mu_1((p-1)(xv)_x + (2-p)v) + \mu_3 v_x.$$

Moreover, the functions g and σ can be obtained from the *reconstruction equations* (34a) and (34b). With the help of (11) and (18) these take the explicit form

$$(43b) \quad g' = \begin{pmatrix} \alpha \\ b \end{pmatrix}' = T_1 L_{(\alpha, b)} \begin{pmatrix} \mu_1 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \alpha \mu_1 \\ \alpha \mu_3 \end{pmatrix}, \quad \alpha(0) = 1, b(0) = 0,$$

$$(43c) \quad \dot{\sigma} = m(g(\sigma)) = m(\alpha(\sigma), b(\sigma)) = \alpha(\sigma)^{2-2p}, \quad \sigma(0) = 0.$$

From Theorem 4.2 (ii) then follows that if $v \in \mathcal{C}([0, \widehat{T}); Y_1) \cap \mathcal{C}^1([0, \widehat{T}); X)$, a solution of the original equation (8) is obtained by formula (33).

- (ii) For the case $d = 2$ we proceed similar and obtain by using the generators calculated in Example 3.5 (ii) the co-moving equation

$$(44a) \quad v_\tau = \nu \Delta v - \frac{1}{p}(|v|^p)_x + \mu_1(p-1)((xv)_x + (yv)_y) + \mu_1(3-2p)v + \mu_3^1 v_x + \mu_3^2 v_y.$$

Moreover, the reconstruction equations (34a) and (34b) become

$$(44b) \quad g' = \begin{pmatrix} \alpha \\ b \end{pmatrix}' = T_1 L_g \begin{pmatrix} \mu_1 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \alpha \mu_1 \\ \alpha \mu_3 \end{pmatrix}, \quad \alpha(0) = 1, b(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$(44c) \quad \dot{\sigma} = m(g(\sigma)) = \alpha(\sigma)^{2-2p}, \quad \sigma(0) = 0.$$

- (iii) Similar considerations yield for $d = 3$ the co-moving equation

$$(45a) \quad v_\tau = \nu \Delta v - \frac{1}{p}(|v|^p)_x + \mu_1(p-1)((xv)_x + (yv)_y + (zv)_z) + \mu_1(4-3p)v \\ + \mu_2^1((yv)_z - (zv)_y) + \mu_3^1 v_x + \mu_3^2 v_y + \mu_3^3 v_z$$

and the reconstruction equations

$$(45b) \quad g' = \begin{pmatrix} \alpha \\ Q \\ b \end{pmatrix}' = T_1 L_g \begin{pmatrix} \mu_1 \\ \mu_2^1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \alpha \mu_1 \\ \mu_2^1 Q \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \alpha \mu_3 \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ Q \\ b \end{pmatrix}(0) = \begin{pmatrix} 1 \\ I \\ 0 \end{pmatrix},$$

$$(45c) \quad \dot{\sigma} = m(g(\sigma)) = \alpha(\sigma)^{2-2p}, \quad \sigma(0) = 0.$$

Note, that $Q \in \text{SO}(2)$ is of the form $Q = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$, so that the differential equation (45b) for Q yields the equations

$$\frac{d}{d\tau} \cos(\phi) = -\sin(\phi)\phi' = -\mu_2^1 \sin(\phi), \quad \frac{d}{d\tau} \sin(\phi) = \cos(\phi)\phi' = \mu_2^1 \cos(\phi).$$

Therefore, $\phi' = \mu_2^1$ completely describes the evolution of the Q -component of g and using this equation, it is implicit that the solution to (45b) always stays on the Lie-group G .

4.2. Phase Conditions. Theorem 4.2 relates the Cauchy-problem for (2)

$$(46) \quad \begin{cases} u_t = \nu \Delta u - \frac{1}{p} \frac{\partial}{\partial x_1} (|u|^p) =: F(u), \\ u(0) = u_0, \end{cases}$$

to the Cauchy-problem in the new, time-dependent coordinate system. Roughly speaking, we can rephrase the result as follows:

A function $u \in \mathcal{C}([0, T]; Y_1) \cap \mathcal{C}^1([0, T]; X)$ solves (46) if and only if the functions $v \in \mathcal{C}([0, \hat{T}]; Y_1) \cap \mathcal{C}^1([0, \hat{T}]; X)$, $\mu \in \mathcal{C}([0, \hat{T}]; \mathfrak{g})$, $g \in \mathcal{C}^1([0, \hat{T}]; G)$, $\sigma \in \mathcal{C}^1([0, T]; [0, \hat{T}])$ solve the system

$$(47) \quad \begin{aligned} v_\tau &= F(v) - T_1 av[\mu(\tau)], & v(0) &= u_0, \\ g_\tau &= T_1 L_{g(\tau)}[\mu(\tau)], & g(0) &= \mathbb{1}, \\ (\sigma^{-1})_\tau &= \frac{1}{m(g(\tau))}, & \sigma(0) &= 0 \end{aligned}$$

and the functions u and v, g, σ are related by the identity $u(t) = a(g(\sigma(t)))v(\sigma(t))$.

Although (46) is a well-posed problem, e.g. in \mathbb{R}^3 for initial data in H^2 (see Remark 2.2), the system (47) is not well-posed, due to an ambiguity in the choice of $\mu(\tau)$. More precisely, the variable μ introduces $\dim \mathfrak{g}$ additional degrees of freedom to the system. As is standard in the numerical freezing method, see e.g. [5], we therefore introduce $\dim \mathfrak{g}$ additional algebraic equations, so called phase conditions, to cope with this ambiguity. In this article we restrict to two specific choices of phase conditions, which work very well in our numerical experiments in Section 5. We will actually use integral phase conditions which were originally introduced by [7] for the numerical approximation of periodic orbits. Thus we are limited to the case $X = L^2$ with $Y_1 = H_{\text{scal}}^2$, and we assume the well-posedness of the problem in these spaces from now on. We note in passing that by using weighted integrals, it is not difficult to allow similar phase conditions also for the $X = \mathcal{C}_0^0$ and $Y_1 = \mathcal{C}_{\text{scal}}^2$ case.

Type 1: Orthogonal phase condition

The idea of the orthogonal phase condition is, to require that the time-evolution of v is always L^2 -orthogonal to its group orbit. This amounts to requiring

$$(48) \quad 0 = \langle v_\tau, T_1 av[\epsilon_j] \rangle, \quad j = 1, \dots, \dim \mathfrak{g},$$

where $\{\epsilon_1, \dots, \epsilon_{\dim \mathfrak{g}}\}$ is a basis of \mathfrak{g} . Inserting the v -equation of (47) into (48) yields

$$(49) \quad 0 = \langle T_1 av[\epsilon_j], F(v) - T_1 av[\mu] \rangle =: \Psi_j^{\text{orth}}(v, \mu), \quad j = 1, \dots, \dim \mathfrak{g},$$

which we call the “*orthogonal phase condition*” and, in fact, is a linear equation for the algebraic variable μ if v is known. Assuming the invertibility of the matrix $\left(\langle T_1 av[\epsilon_j], T_1 av[\epsilon_k] \rangle_{j,k=1 \dots \dim \mathfrak{g}} \right) \in \mathbb{R}^{\dim \mathfrak{g}, \dim \mathfrak{g}}$, (49) can easily be solved for the algebraic variables μ ,

$$\mu = \left(\langle T_1 av[\epsilon_j], T_1 av[\epsilon_k] \rangle_{j,k=1 \dots \dim \mathfrak{g}} \right)^{-1} \langle T_1 av[\epsilon_j], F(v) \rangle_{j=1 \dots \dim \mathfrak{g}}.$$

In principle it is possible to insert this formula directly into (47), but we rather supplement (47) with (49), since we are anyway also interested in the actual value of μ as an important constant of motion, as already argued in the introduction.

Type 2: Fixed phase condition

The idea of the fixed phase condition is, to require that the v -component of the solution always lies in a fixed, $\dim \mathfrak{g}$ -co-dimensional hyperplane, which is given as the level set of a fixed, linear mapping, i.e.

$$(50) \quad 0 = \psi_j(v) - r_j =: \Psi_j^{\text{fix}}(v), \quad j = 1, \dots, \dim \mathfrak{g},$$

where $\psi_j \in X^*$, $j = 1, \dots, \dim \mathfrak{g}$, are linearly independent elements of the dual of X and $r_j \in \mathbb{R}$. A standard choice, cf. [5], is the following: Assume that there is a “suitable” reference function \hat{u} given and then one require that the v -component of the solution always satisfies

$$(51) \quad 0 = \langle T_1 a \hat{u}[\epsilon_j], v - \hat{u} \rangle, \quad j = 1, \dots, \dim \mathfrak{g},$$

where $\{\epsilon_1, \dots, \epsilon_{\dim \mathfrak{g}}\}$ is a basis of \mathfrak{g} .

Remark 4.7. The phase condition (51) can also be obtained by requiring that the v -component of the solution is always better aligned in the L^2 -norm to \hat{u} than to any other element of the group orbit of \hat{u} , i.e.

$$(52) \quad \operatorname{argmin}_{g \in G} \|a(g)\hat{u} - v\|_{L^2}^2 = 1.$$

A necessary condition for (52) is (51).

We now augment system (47) with one of the phase conditions (49) or (51) and obtain the PDAE system of the numerical freezing method

$$(53a) \quad v_\tau = F(v) - T_1 av[\mu(\tau)], \quad v(0) = u_0,$$

$$(53b) \quad 0 = \Psi(v, \mu),$$

$$(53c) \quad g_\tau = T_1 L_{g(\tau)}[\mu(\tau)], \quad g(0) = 1,$$

$$(53d) \quad (\rho)_\tau = \frac{1}{m(g(\tau))}, \quad \rho(0) = 0,$$

where $\Psi(v, \mu)$ is either Ψ^{orth} from (49) or Ψ^{fix} from (50), and we denote $\rho = \sigma^{-1}$.

- Remarks 4.8.** (1) Note that the ordinary differential equations (53c) and (53d) actually decouple from (53a) and (53b) and hence could be solved in a post-processing step.
- (2) Observe that (53) consists of the PDE (53a) which has a hyperbolic-parabolic structure, coupled to a system of ordinary differential equations (53c) and (53d) and coupled to a system of algebraic equations (53b). Moreover, in (53a) the hyperbolic part dominates for $\nu \ll 1$ and the parabolic part dominates for ν sufficiently large.
- (3) For the choice (49) the system (53a), (53b) is a partial differential algebraic equation of “time-index” 1 and for the choice (51) it is of “time-index” 2. Here we understand the index as a differentiation index (see [12]).

5. NUMERICAL RESULTS

In this section we now present the result of several numerical experiments. We use a numerical second order scheme that we develop in [17]. Here we do not go into the details of the numerical scheme but only mention that it is based on a central method-of-lines system for hyperbolic conservation laws, adapted from [11] to (53), and then fully discretized with an IMEX-Runge-Kutta time-discretization in the spirit of [1] to cope with the different parts of the equation. For details we refer to [17].

To distinguish between the original coordinates and the coordinates of the freezing method, we always denote original space by $x \in \mathbb{R}^d$ and the original time by t and the original solution in these coordinates is denoted by u , whereas $\xi \in \mathbb{R}^d$ and $\tau \geq 0$ denote the space and time in the new coordinates and v is the solution in these new coordinates.

5.1. 1d-Experiments. We choose the spatial step size $\Delta\xi = 0.01$ and the time step size $\Delta\tau$ is a CFL-based multiple of $\Delta\xi$, see [17]. In all 1d-experiments we choose the initial condition given by

$$(54) \quad u_0(x) = \begin{cases} \sin(2x), & -\frac{\pi}{2} \leq x \leq 0, \\ \sin(x), & 0 \leq x \leq \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this belongs to $H^1(\mathbb{R})$ but not to $H^2(\mathbb{R})$. In our experiments we observed that the method and our numerical scheme work just well also for piecewise continuous initial data. Moreover, if nothing else is stated, we choose the fixed phase condition, where we choose the initial condition as a suitable reference function. In case the solution evolves too far away from this reference function, we update it with the current state of the solution.

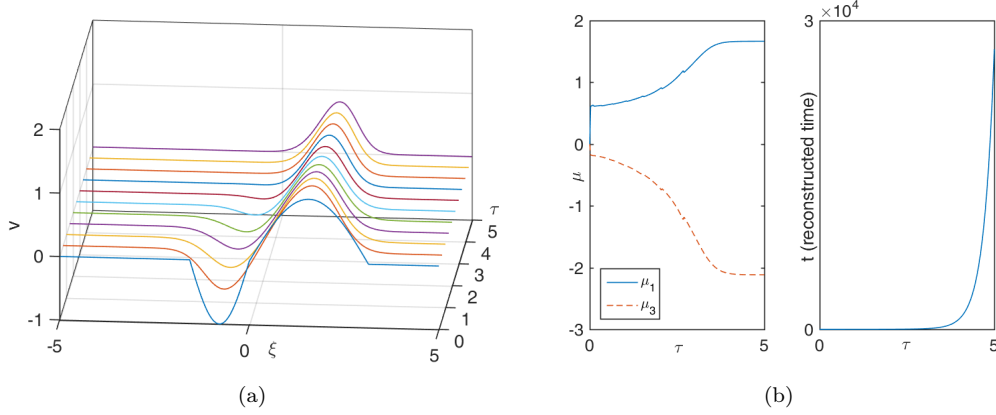


FIGURE 1. Plots of the time evolution for conservative 1d-Burgers equation ($p = 2$) with viscosity $\nu = 0.4$ (a) in the scaled (computational) coordinates at different time instances, (b) the evolution of the variables in the Lie-algebra and the evolution of the original time as function of the scaled time $t(\tau)$.

Variation of the parameter p . In our first series of computations we consider the behavior of the freezing method for the Cauchy-problem for Burgers' equation

$$u_t = 0.4u_{xx} - \frac{1}{p}\partial_x(|u|^p), \quad u(0) = u_0$$

where we choose different values for the parameter p . In Fig. 1 we show the results for the freezing method for the conservative 1d Burgers' equation (i.e. $p = 2$). One can very well observe that the solution to the freezing PDAE (53a), (53b) stabilizes as τ increases. Moreover, also the algebraic variables μ_1 (scaling) and μ_3 (spatial velocity) converge to constant values as τ increases. We also calculate the solution to the reconstruction equations (53c), (53d) and use these to obtain the solution u in the original coordinates, given by formula (33) and shown Fig. 2. The time-instances are precisely $t = t(\tau)$ with the τ from

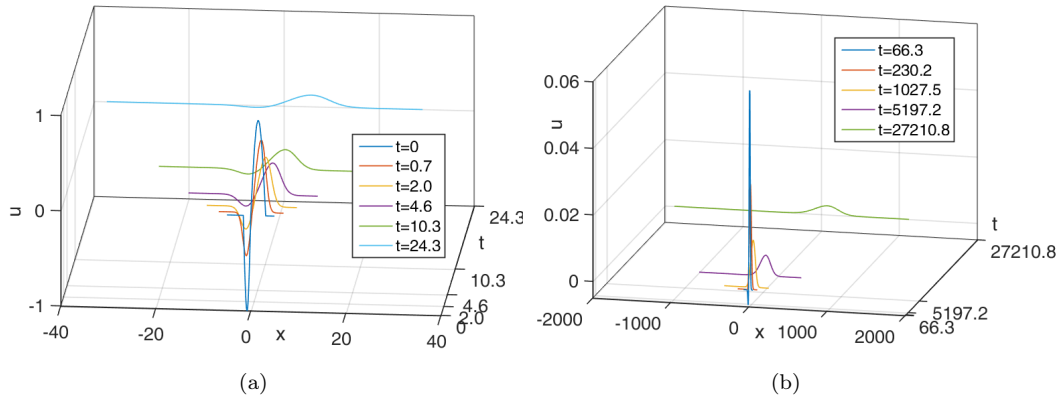


FIGURE 2. Reconstructed solutions of the 1d-Burgers' equation ($p = 2$) with viscosity $\nu = 0.4$. Initial time interval $[0, 24.3]$ (a) and later time interval $[66.3, 27210.8]$ (b).

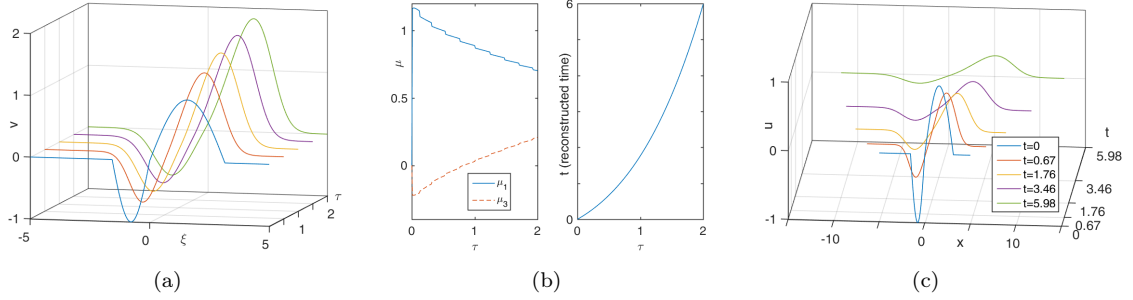


FIGURE 3. Time evolution of 1d-Burgers' equation with $p = \frac{3}{2}$ and viscosity $\nu = 0.4$. In (a) several time-instances of the numerical solution of the freezing method are shown, (b) shows how the algebraic variables μ_1 (scaling) and μ_3 (spatial velocity) depend on the scaled time τ , and (c) shows the solutions from (a) in the original coordinates.

Fig. 1(a). As is well-known, in the original coordinates the solution stabilizes to the constant zero as time tends to infinity. Note that the original time $t(\tau)$ increases very rapidly with τ (e.g. $t(5) \approx 27210.8$) and one needs a very large time-domain if the solution is calculated in the original coordinates.

We repeat the above numerical experiment for the Cauchy problem

$$u_t = u_{xx} - \frac{2}{3} \partial_x (|u|^{\frac{3}{2}}), \quad u(0) = u_0,$$

where we changed the parameter p to $p = \frac{3}{2}$. The result of this computation is presented in Fig. 3. One can nicely observe that the mass of the profile v in the new coordinates grows as τ increases, which is in accordance with Remark 4.4 and formula (42) since $p < \frac{d+1}{d}$ and $\mu_1 > 0$. Nevertheless, we can still do the freezing method calculations on a fixed bounded domain and obtain the solution in the original coordinates from the reconstruction equations. The result is shown in Fig. 3(c).

We also repeat the experiment with $p = \frac{5}{2}$. From Remark 4.4 and formula (42) we now expect that the profile v in the co-moving coordinates decays to zero as τ tends to infinity. The results of the simulation with the numerical freezing method are shown in Fig. 4 and precisely reproduce this expectation. Note that we actually only calculated until $\tau = 4$ in the scaled coordinates. But $\tau = 4$ corresponds to $t \approx 341$,

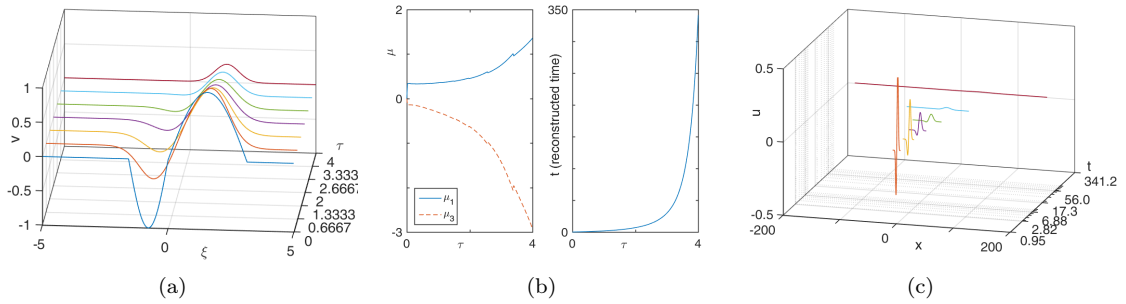


FIGURE 4. Time evolution for the generalized 1d-Burgers equation with $p = \frac{5}{2}$ and viscosity $\nu = 0.4$. In (a) the solution of the freezing method at different time-instances is shown, (b) shows the algebraic variables μ and original time t plotted as functions of τ . Finally, (c) shows the solution from (a) in the original coordinates.

and in the original coordinates the profile $v(\tau = 4)$, which is calculated on the fixed domain $\xi \in [-5, 5]$ corresponds to the solution in the original coordinates on the domain $x \in [-187.44, 151.54]$. Also note that we have chosen a logarithmic scale for the time-axis in Fig. 4(c). We again see that, although there is no true relative equilibrium of the equation $u_t = u_{xx} - \frac{2}{5}\partial_x(|u|^{\frac{5}{2}})$, the freezing method allows us to do a calculation on a fixed bounded domain for a far longer time, than it would be possible for the original problem.

Different phase conditions. In all the experiments performed so far, we have chosen the fixed phase condition given by (51). We now compare the results we obtain with the freezing method by using the fixed phase condition with the results we obtain when we use the orthogonal phase condition. We again

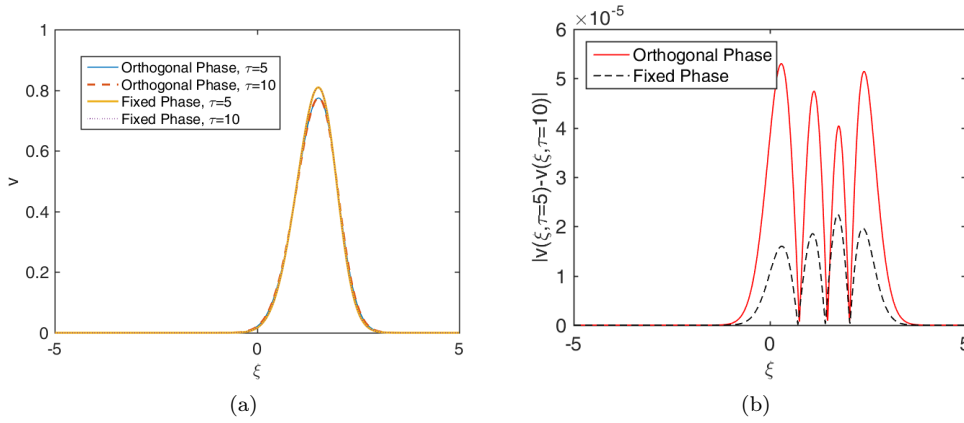


FIGURE 5. Comparison of the final states for different phase conditions. In (a) the solutions obtained with the freezing method for the orthogonal and fixed phase conditions at $\tau = 5$ and $\tau = 10$ are shown. The solutions for the same phase conditions do not change from $\tau = 5$ to $\tau = 10$. In (b) we plot the actual difference of $|v(\xi, 5) - v(\xi, 10)|$.

consider the conservative Burgers' equation, i.e. $p = 2$, and choose the viscosity $\nu = 0.4$. In Fig. 5 we show the solution to the freezing method obtained with the fixed phase condition at $\tau = 5$ and $\tau = 10$ and also the solution obtained with the orthogonal phase condition at the same time-instances $\tau = 5$ and $\tau = 10$. In Fig. 5(a) these are plotted in one diagram and the solutions with the same phase conditions but at different time-instances virtually do not differ and, therefore, seem to be constant rest states. The difference of the solutions to the same phase condition but at different times is shown in Figure 5(b). As is obvious from Fig. 5(a), these steady states do depend on the choice of the phase condition and, moreover, we even obtain different limits for the algebraic values. In the fixed phase condition case we obtain $\mu_1 = 1.665$ (scaling) and $\mu_3 = -2.117$ (translation), in the orthogonal phase condition case we obtain $\mu_1 = 1.524$ and $\mu_3 = -1.931$. Nevertheless, the solution in the original coordinates at the latest common original time is plotted in Fig. 6 and shows only a very small difference between the two different choices of phase conditions as plotted in Fig. 6(b) and Fig. 6(c).

Metastable behavior. Our method directly enables us to observe the metastable behavior in Burgers' equation with small viscosity which was first numerically observed and analyzed in [10] and later discussed from a dynamical systems point of view in [3]. Our results are presented in Fig. 7 for viscosity $\nu = 0.01$. Note that there is a very long transient, when the solution pretty much looks like an N-wave (see Fig. 7(a) until $\tau \approx 200 \doteq t \approx 10^{41}$) and then in a final stage converges to the true similarity solution, which is a viscosity wave. This convergence can also be observed by looking at the time evolution of the algebraic variable μ , which evolves slowly first and finally stabilizes to a constant value.

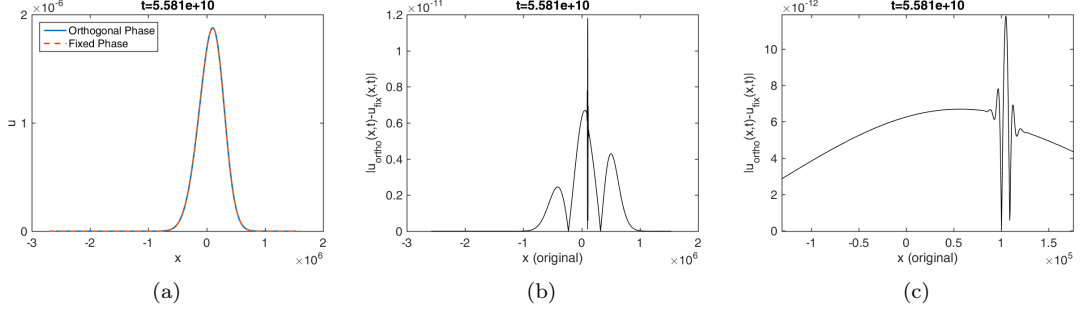


FIGURE 6. Difference of the solutions obtained at $t = 5.581 \cdot 10^{10}$ via the freezing method with orthogonal and fixed phase condition. In (a) the two different solutions are shown, (b) and (c) shows there difference.

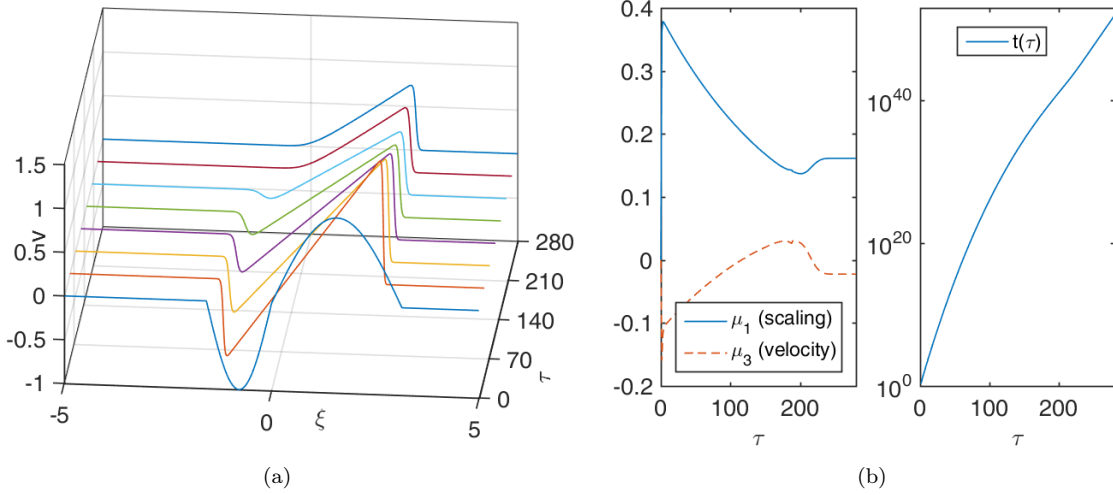


FIGURE 7. Time evolution for 1d-Burgers' equation, with viscosity $\nu = 0.01$ in the scaled (computational) coordinates (a) and the evolution of the algebraic variables μ and of the original time as function of the scaled time $t(\tau)$.

5.2. 2d-Experiments. We also apply the method to the two-dimensional generalized Burgers' equations

$$\partial_t u = \nu \Delta u - \frac{1}{p} \partial_x (|u|^p).$$

Again we use the numerical scheme introduced in [17]. Moreover, we choose the orthogonal phase condition, which is slightly more efficient than the fixed phase condition.

Metastable behavior. First we consider the conservative 2-dimensional Burgers' equation, i.e. $p = \frac{3}{2}$ with very small viscosity $\nu = 0.05$. For the actual computation with the freezing method we choose the computational domain $\xi \in [-5, 5] \times [-5, 5]$ and no-flux boundary conditions. The spatial step sizes are $\Delta \xi_1 = \Delta \xi_2 = \frac{1}{15}$ and the time step size $\Delta \tau$ is CFL-base multiple of these, see [17]. In Fig. 8(a) we present contour plots of the solution at different time instances and one observes that rapidly a pattern evolves which resembles the 1d pattern of an N-wave. This pattern exists for a very long time until the

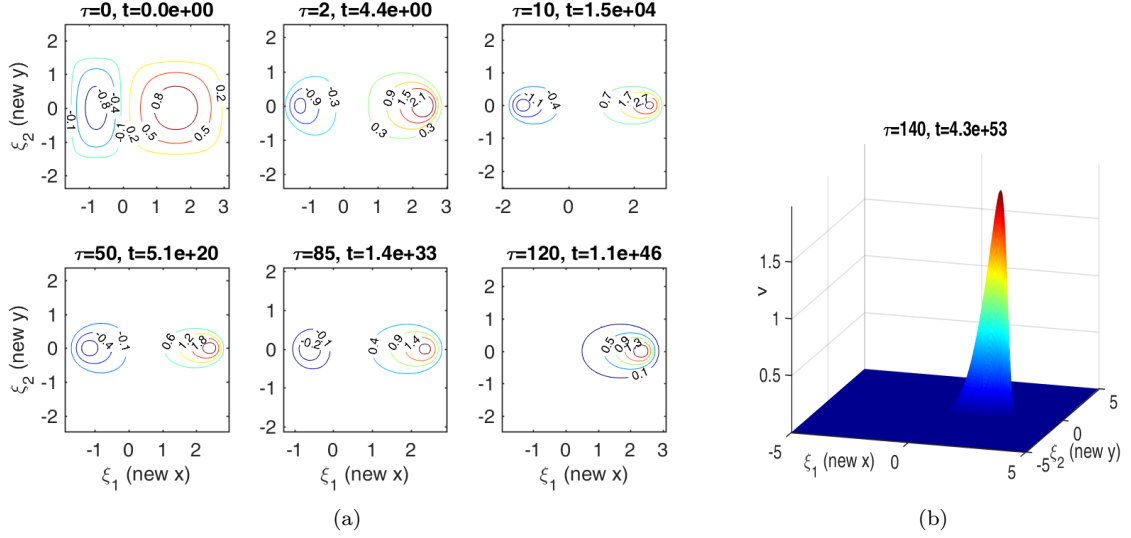


FIGURE 8. Time Evolution of the 2d-Burgers' Equation in co-moving coordinates for $\nu = 0.05$. In (a) we plot the contour lines at different time instances (in the scaled time) and in (b) we show the solution at the final time $\tau = 140$ ($\hat{=} t \approx 4.3 \cdot 10^{53}$).

“negative blob” vanishes and a final steady state is reached. This happens approximately at $\tau = 110$ ($\hat{=} t = 1.69 \cdot 10^{42}$). A plot of this final state is shown in Fig. 8(b). Fig. 9 shows that this profile indeed is a steady state: The difference of the solution to the freezing method at $\tau = 80$ ($\hat{=} t = 2.60 \cdot 10^{31}$) and $\tau = 110$ ($\hat{=} t = 1.69 \cdot 10^{42}$) is plotted in Fig. 9(a). Note that there is a large negative area where the solutions differ the most (actually their values differ by approximately -0.37). This area corresponds to the “negative blob” which vanishes as τ increases (Fig. 8(a)). When comparing the solutions at $\tau = 110$ ($\hat{=} t = 1.69 \cdot 10^{42}$) and $\tau = 140$ ($\hat{=} t = 4.27 \cdot 10^{53}$), this spot indeed has vanished and the difference of

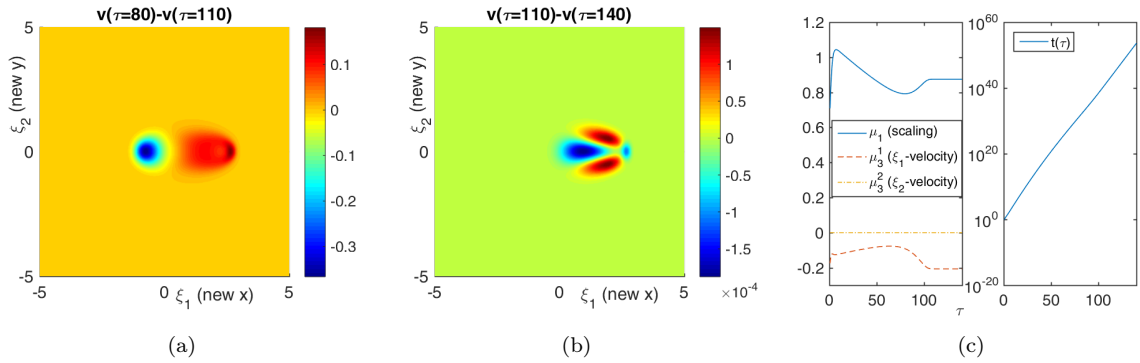


FIGURE 9. Convergence to relative equilibrium: In (a) and (b) we plot the difference of the solutions to the freezing equations for different times τ . In (c) the evolution of the evolution of the algebraic variables and the original time is shown.

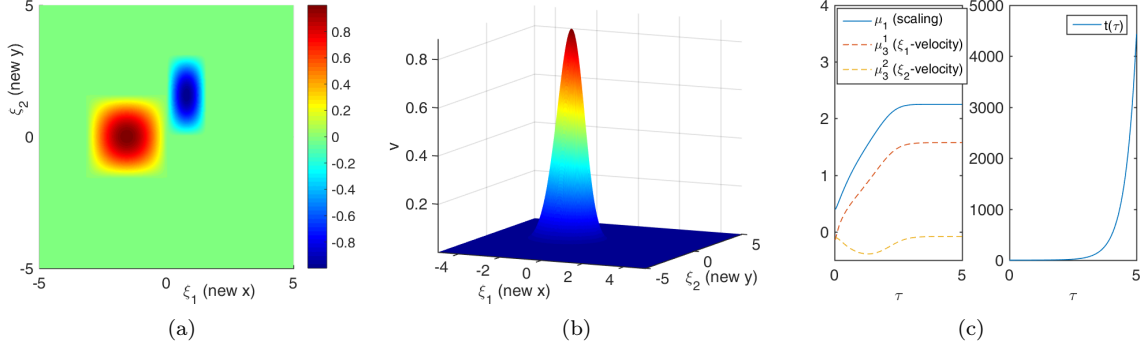


FIGURE 10. Freezing method for $u_t = 0.4 \Delta u - \frac{2}{3} \partial_x (|u|^{\frac{3}{2}})$: (a) shows the initial condition, (b) the steady state obtained at $\tau = 5$, and (c) shows the evolution of the algebraic variables and original time as functions of τ .

these two solutions is only of order 10^{-4} . Also the algebraic variables, shown in Fig. 9(c), slowly evolve until $\tau \approx 110$, when they finally stabilize.

Variation of the parameter p . We finally consider the long-time behavior of Burgers' equation with fixed viscosity $\nu = 0.4$ and vary the parameter p . For the experiments we choose the initial condition

$$u_0(x, y) = \begin{cases} \sin(2x) \sin(y), & -\frac{\pi}{2} < x < 0, 0 < y < \pi, \\ \sin(x) \cos(y), & 0 < x < \pi, -\frac{\pi}{2} < y < \frac{\pi}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

which is depicted in Fig. 10(a).

For the conservative Burgers' equation, i.e. $p = \frac{3}{2}$, we again observe the convergence to a steady state of the freezing equations and hence to a relative equilibrium. In Figure 10 we plot the final state of the calculation at $\tau = 5$ which corresponds to $t \approx 4436.7$. Note that the final state has a non-constant ξ_2 -velocity, which is due to the fact that the function does not have a center of mass along the ξ_1 -axis. For the standard 2d-Burgers' equation, that is $p = 2$, we obtain that the solution (v, μ) of the freezing method does not converge to a localized steady state but the total mass of v decays as predicted by Remark 4.4. Nevertheless, it is possible and does make sense to compute the solution to the original equation by using the freezing method on a fixed computational domain. The final state at $\tau = 5$ ($\hat{=t} = 11$) and its reconstruction to the original coordinates are shown in Fig. 11.

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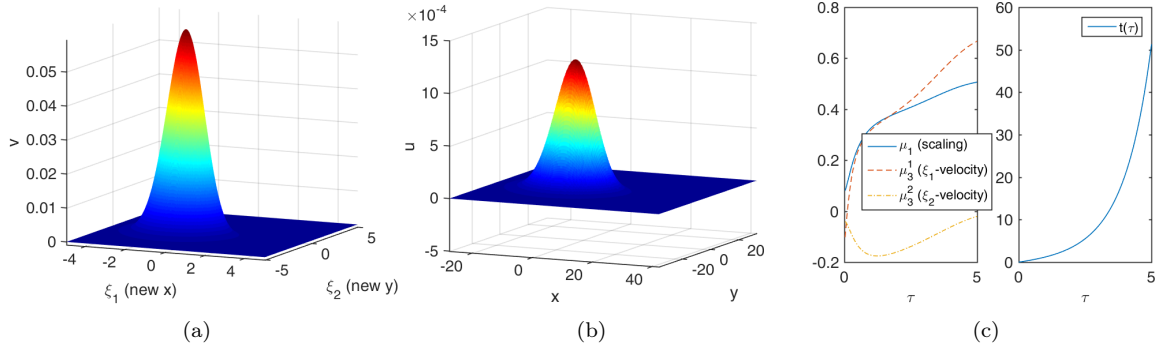


FIGURE 11. Solution of the freezing method at $\tau = 5$ for the 2d-Burgers' equation $u_t = 0.4 \Delta u - \frac{1}{2} \partial_x (|u|^2)$. In (a) in the computational coordinates, in (b) reconstruction of the solution, and evolution of the algebraic variables in (c).

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